

SIMULTANEOUS GENERATION FOR ZETA VALUES BY THE MARKOV-WZ METHOD

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ABSTRACT. By application of the Markov-WZ method, we prove a more general form of a bivariate generating function identity containing, as particular cases, Koecher's and Almkvist-Granville's Apéry-like formulae for odd zeta values. As a consequence, we get a new identity producing Apéry-like series for all $\zeta(2n + 4m + 3)$, $n, m \geq 0$, convergent at the geometric rate with ratio 2^{-10} .

1. INTRODUCTION

The Riemann zeta function is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{for } \operatorname{Re}(s) > 1.$$

Apéry's irrationality proof of $\zeta(3)$ [13] operates with the faster convergent series

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}$$

first obtained by A. A. Markov in 1890 [10]. The general formula giving analogous series for all $\zeta(2s + 3)$, $s \geq 0$, was proved by Koecher [7] (and independently in an expanded form by Leshchiner [9])

$$(1) \quad \sum_{s=0}^{\infty} \zeta(2s + 3)x^{2s} = \sum_{k=1}^{\infty} \frac{1}{k(k^2 - x^2)} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \frac{5k^2 - x^2}{k^2 - x^2} \prod_{m=1}^{k-1} \left(1 - \frac{x^2}{m^2}\right).$$

The similar identity generating fast convergent series for all $\zeta(4s + 3)$, $s \geq 0$, which for $s > 1$ are different from Koecher's result (1) was experimentally discovered in [3] and proved by G. Almkvist and A. Granville in [1]

$$(2) \quad \sum_{s=0}^{\infty} \zeta(4s + 3)x^{4s} = \sum_{k=1}^{\infty} \frac{k}{k^4 - x^4} = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\binom{2k}{k}} \frac{k}{k^4 - x^4} \prod_{m=1}^{k-1} \left(\frac{m^4 + 4x^4}{m^4 - x^4}\right).$$

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There exists a bivariate unifying formula for identities (1) and (2)

$$(3) \quad \sum_{k=1}^{\infty} \frac{k}{k^4 - x^2 k^2 - y^4} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k \binom{2k}{k}} \frac{5k^2 - x^2}{k^4 - x^2 k^2 - y^4} \prod_{m=1}^{k-1} \left(\frac{(m^2 - x^2)^2 + 4y^4}{m^4 - x^2 m^2 - y^4} \right),$$

which was first conjectured by H. Cohen and then proved by D. Bradley [5] and, independently, by T. Rivoal [14]. This identity implies (1) if $y = 0$, and gives (2) if $x = 0$. The proof of (3) relies on Borwein & Bradley's method [3] and consists of reduction of (3) to a finite non-trivial combinatorial identity which can be proved on the base of Almkvist & Granville's work [1].

In this short note, we prove a more general form of (3) by application of the Markov-WZ method (see [10, 12, 11, 8]). Let us notice that Koecher's identity (1) and similar Leschiner's and Bailey-Borwein-Bradley's identities [9, 4] generating rapidly convergent series for even zeta values $\zeta(2s+2)$ can be proved with the help of the WZ method (see [6] for more details).

Theorem 1. *Let a, b be complex numbers, with $|a| < 1$, $|b| < 1$. Then for arbitrary complex numbers A_0, B_0, C_0 we have*

$$\sum_{k=1}^{\infty} \frac{A_0 + B_0 k + C_0 k^2}{(k^2 - a^2)(k^2 - b^2)} = \sum_{n=1}^{\infty} \frac{d_n}{\prod_{m=1}^n (m^2 - a^2)(m^2 - b^2)},$$

with

$$d_n = \frac{(-1)^{n-1} B_0 (n-1)! (5n^2 - a^2 - b^2)}{2^{n+1}} \prod_{m=1}^{n-1} \left(\frac{(m^2 - (a^2 + b^2))^2 - 4a^2 b^2}{2m + 1} \right) \\ + \frac{20n + 5}{2(5n^2 - 2a^2 - 2b^2)} L_n + \frac{35n^5 - 35n^3(a^2 + b^2) + 4n(3a^4 + 3b^4 - 4a^2 b^2)}{4(5n^2 - 2a^2 - 2b^2)} L_{n-1},$$

where L_n is a solution of the second order difference equation

$$4(4n + 3)(4n + 5)(5n^2 - 2a^2 - 2b^2)L_{n+1} + 2(n + 1)p(n)L_n \\ - n(n + 1)(5(n + 1)^2 - 2a^2 - 2b^2)q(n)L_{n-1} = 0, \quad n = 1, 2, \dots$$

with initial conditions $L_0 = C_0$,

$$L_1 = \left(\frac{1}{3} - \frac{2}{15}(a^2 + b^2) \right) A_0 + \left(\frac{1}{6}(a^2 + b^2) - \frac{2}{15}(a^4 + b^4 - 4a^2 b^2) - \frac{1}{30} \right) C_0,$$

and

$$(4) \quad p(n) = 30n^7 + 105n^6 + n^5(145 - 52(a^2 + b^2)) + n^4(100 - 130(a^2 + b^2)) \\ + n^3(35 - 124(a^2 + b^2) + 56(a^4 + b^4) - 208a^2 b^2) + n^2(5 - 56(a^2 + b^2) \\ + 84(a^4 + b^4) - 312a^2 b^2) + n(80a^2 b^2(a^2 + b^2) - 16(a^6 + b^6) + 48(a^4 + b^4 - 3a^2 b^2) \\ - 14(a^2 + b^2)) + (10(a^2 - b^2)^2 - 2(a^2 + b^2) + 40a^2 b^2(a^2 + b^2) - 8(a^6 + b^6)), \\ (5) \quad q(n) = n^8 - 6n^6(a^2 + b^2) + n^4(9(a^4 + b^4) + 30a^2 b^2) \\ - n^2(28a^2 b^2(a^2 + b^2) + 4(a^6 + b^6)) + 16a^2 b^2(a^2 - b^2)^2.$$

If in Theorem 1 we take $B_0 = 1$, $A_0 = C_0 = 0$, then $L_n = 0$ for all $n \geq 0$ and putting

$$(6) \quad a^2 = \frac{x^2 + \sqrt{x^4 + 4y^4}}{2}, \quad b^2 = \frac{x^2 - \sqrt{x^4 + 4y^4}}{2}$$

we get the identity (3).

If $A_0 = 1$, $B_0 = C_0 = a = b = 0$, we get the following series for $\zeta(4)$ mentioned by Markov in [10, p.18]:

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n!^4} \left(\frac{4n+1}{2n^2} L_n + \frac{7n^3}{4} L_{n-1} \right),$$

where $L_0 = 0$, $L_1 = 1/3$, and

$$4(4n+3)(4n+5)L_{n+1} + 2(n+1)^3(6n^3+9n^2+5n+1)L_n - n^7(n+1)^3L_{n-1} = 0, \quad n \geq 1.$$

Theorem 2. *Let x, y be complex numbers such that $|x|^2 + |y|^4 < 1$. Then*

$$(7) \quad \sum_{k=1}^{\infty} \frac{k}{k^4 - x^2k^2 - y^4} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}r(n)}{n \binom{2n}{n}} \frac{\prod_{m=1}^{n-1} ((m^2 - x^2)^2 + 4y^4)}{\prod_{m=n}^{2n} (m^4 - x^2m^2 - y^4)},$$

where

$$r(n) = 205n^6 - 160n^5 + (32 - 62x^2)n^4 + 40x^2n^3 + (x^4 - 8x^2 - 25y^4)n^2 + 10y^4n + y^4(x^2 - 2).$$

Since

$$\sum_{k=1}^{\infty} \frac{k}{k^4 - x^2k^2 - y^4} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{n+m}{n} \zeta(2n+4m+3) x^{2n} y^{4m},$$

the formula (7) generates Apéry-like series for all $\zeta(2n+4m+3)$, $n, m \geq 0$, convergent at the geometric rate with ratio 2^{-10} . So, for example, if $x = y = 0$, we get Amdeberhan and Zeilberger's series [2] for $\zeta(3)$

$$\zeta(3) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(205n^2 - 160n + 32)}{n^5 \binom{2n}{n}^5}.$$

If $y = 0$, we recover Theorem 4 from [6]. If $x = 0$, we find, in particular, the following expression for $\zeta(7)$:

$$\begin{aligned} \zeta(7) &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n(25n^2 - 10n + 2)}{n^9 \binom{2n}{n}^5} \\ &\quad - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n(205n^2 - 160n + 32)}{n^5 \binom{2n}{n}^5} \left(\sum_{m=1}^{2n} \frac{1}{m^4} + \sum_{m=1}^{n-1} \frac{3}{m^4} \right). \end{aligned}$$

2. PROOF OF THEOREM 1.

As usual, let $(\lambda)_\nu$ be the Pochhammer symbol (or the shifted factorial) defined by

$$(\lambda)_\nu = \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1, & \nu = 0; \\ \lambda(\lambda + 1) \dots (\lambda + \nu - 1), & \nu \in \mathbb{N}. \end{cases}$$

Let a, b be complex numbers such that $|a| < 1$, $|b| < 1$. We start with the kernel

$$H(n, k) = \frac{(1+a)_k(1-a)_k(1+b)_k(1-b)_k}{(1+a)_{n+k+1}(1-a)_{n+k+1}(1+b)_{n+k+1}(1-b)_{n+k+1}}$$

and define two functions

$$\begin{aligned} F(n, k) &= H(n, k)(A(n) + B(n)(k+1) + C(n)(k+1)^2), \\ G(n, k) &= H(n, k)(D(n) + E(n)k + K(n)k^2 + L(n)k^3), \end{aligned}$$

with 7 unknown coefficients $A(n), B(n), C(n), D(n), E(n), K(n), L(n)$ as functions of n . We require that

$$(8) \quad F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k).$$

Since $F(n, k)$ and $G(n, k)$ are not proper hypergeometric, the pair (F, G) is a Markov WZ-pair (see [12, p. 8], [15] for definitions).

Substituting F, G into (8) and cancelling common factors we get the following equation of degree 6 in a variable k :

$$\begin{aligned} & ((n+k+2)^2 - a^2)((n+k+2)^2 - b^2)(A(n) + B(n)(k+1) + C(n)(k+1)^2) \\ & - A(n+1) - B(n+1)(k+1) - C(n+1)(k+1)^2 = ((n+k+2)^2 - a^2) \\ (9) \quad & \times ((n+k+2)^2 - b^2)(D(n) + E(n)k + K(n)k^2 + L(n)k^3) - ((k+1)^2 - a^2) \\ & \times ((k+1)^2 - b^2)(D(n) + E(n)(k+1) + K(n)(k+1)^2 + L(n)(k+1)^3). \end{aligned}$$

To satisfy condition (8), all the coefficients of the powers of $(k+1)$ in the equation (9) must vanish. This leads to a system of first order linear recurrence equations with polynomial coefficients for $A(n), B(n), C(n), D(n), E(n), K(n), L(n)$

$$(10) \quad C(n) = (4n_1 - 3)L(n), \quad B(n) = (4n_1 - 2)K(n) - (10n_1^2 - 3)L(n), \quad n_1 = n + 1,$$

$$(11) \quad A(n) = (4n_1 - 1)E(n) - (10n_1^2 - 1)K(n) + (20n_1^3 + 2n_1(a^2 + b^2) - 1)L(n),$$

$$(12) \quad 4D(n) = 10n_1E(n) - (20n_1^2 + 2a^2 + 2b^2)K(n) + (35n_1^3 + 11n_1(a^2 + b^2))L(n),$$

$$\begin{aligned} (13) \quad 2(4n_1 + 1)L(n+1) &= 2n_1(5n_1^2 - 2a^2 - 2b^2)E(n) - 2n_1^2(15n_1^2 - 6(a^2 + b^2))K(n) \\ &+ n_1(63n_1^4 - 17n_1^2(a^2 + b^2) - 4(a^4 + b^4))L(n), \end{aligned}$$

$$\begin{aligned} (14) \quad & 2(4n_1 + 2)K(n+1) - 2(10n_1^2 + 20n_1 + 7)L(n+1) \\ &= 2n_1^2(5n_1^2 - 2(a^2 + b^2))E(n) - 2(16n_1^5 - 8n_1^3(a^2 + b^2) + n_1(a^2 - b^2)^2)K(n) \\ &+ (70n_1^6 - 31n_1^4(a^2 + b^2) + n_1^2(3a^4 + 3b^4 - 14a^2b^2))L(n), \end{aligned}$$

$$\begin{aligned}
(15) \quad & 4(20(n_1 + 1)^3 + 2(n_1 + 1)(a^2 + b^2) - 1)L(n + 1) - 4(10n_1^2 + 20n_1 + 9)K(n + 1) \\
& + 4(4n_1 + 3)E(n + 1) = (6n_1^5 - 6n_1^3(a^2 + b^2) + 16a^2b^2n_1)E(n) \\
& - (20n_1^6 - 22n_1^4(a^2 + b^2) + 2n_1^2(a^4 + b^4 + 22a^2b^2))K(n) \\
& + (45n_1^7 - 48n_1^5(a^2 + b^2) + n_1^3(3a^4 + 3b^4 + 86a^2b^2) + 8a^2b^2n_1(a^2 + b^2))L(n).
\end{aligned}$$

Multiplying equation (13) by n_1 and subtracting from (14) we get

$$2K(n + 1) - 7(n_1 + 1)L(n + 1) = -\frac{n_1((n_1^2 - a^2 - b^2)^2 - 4a^2b^2)}{2(2n_1 + 1)}(2K(n) - 7n_1L(n)),$$

which yields

$$K(n) - \frac{7}{2}n_1L(n) = \frac{(-1)^n(2K(0) - 7L(0))n!}{2^{n_1}} \prod_{m=1}^n \left(\frac{(m^2 - a^2 - b^2)^2 - 4a^2b^2}{2m + 1} \right).$$

From (10) it follows that $2K(0) = B(0) + 7L(0)$ and therefore,

$$(16) \quad K(n) = \frac{7}{2}n_1L(n) + \frac{(-1)^nB(0)n!}{2^{n_1}} \prod_{m=1}^n \left(\frac{(m^2 - a^2 - b^2)^2 - 4a^2b^2}{2m + 1} \right).$$

Substituting (16) into (13) yields the formula for $E(n)$

$$\begin{aligned}
(17) \quad E(n) &= \frac{4n_1 + 1}{n_1(5n_1^2 - 2a^2 - 2b^2)}L(n + 1) + \frac{42n_1^4 - 25n_1^2(a^2 + b^2) + 4(a^4 + b^4)}{2(5n_1^2 - 2a^2 - 2b^2)}L(n) \\
&+ \frac{3B(0)(-1)^nn_1!}{2^{n_1}} \prod_{m=1}^n \left(\frac{(m^2 - a^2 - b^2)^2 - 4a^2b^2}{2m + 1} \right).
\end{aligned}$$

Substituting (16) and (17) into (12) gives the formula for $D(n)$

$$\begin{aligned}
(18) \quad D(n) &= \frac{(40n_1 + 10)L(n + 1) + (35n_1^5 - 35n_1^3(a^2 + b^2) + 4n_1(3a^4 + 3b^4 - 4a^2b^2))L(n)}{4(5n_1^2 - 2a^2 - 2b^2)} \\
&+ \frac{(-1)^nB(0)n!(5n_1^2 - a^2 - b^2)}{2^{n_1+1}} \prod_{m=1}^n \left(\frac{(m^2 - a^2 - b^2)^2 - 4a^2b^2}{2m + 1} \right).
\end{aligned}$$

Finally, substituting (16), (17) into (15) gives the second-order difference equation for $L(n)$

$$\begin{aligned}
& 4(4n + 3)(4n + 5)(5n^2 - 2a^2 - 2b^2)L(n + 1) + 2(n + 1)p(n)L(n) \\
& - n(n + 1)(5(n + 1)^2 - 2a^2 - 2b^2)q(n)L(n - 1) = 0,
\end{aligned}$$

with initial conditions $L(0) = C(0)$,

$$L(1) = \left(\frac{1}{3} - \frac{2}{15}(a^2 + b^2) \right) A(0) + \left(\frac{1}{6}(a^2 + b^2) - \frac{2}{15}(a^4 + b^4 - 4a^2b^2) - \frac{1}{30} \right) C(0),$$

derived from (10), (11), (13) and polynomials $p(n), q(n)$ defined in (4), (5).

If we put $l(n) = L(n)/(n!)^4$, $n = 0, 1, 2, \dots$, then it is easily seen that the sequence $l(n)$ satisfies the following recurrence equation:

$$4(4n+3)(4n+5)(5n^2-2a^2-2b^2)n^3(n+1)^3l(n+1) + 2n^3p(n)l(n) - (5(n+1)^2-2a^2-2b^2)q(n)l(n-1) = 0.$$

Its characteristic polynomial $64\lambda^2 + 12\lambda - 1 = 0$ has two different zeros $\lambda_1 = -1/4$, $\lambda_2 = 1/16$, and by Poincaré's theorem, we get

$$(19) \quad \lim_{n \rightarrow \infty} \left(\frac{|L(n)|}{(n!)^4} \right)^{\frac{1}{n}} \leq \frac{1}{4}.$$

The limit inequality (19) implies

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} F(n, k) = 0, \quad \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} G(n, k) = 0$$

and therefore, we have (see [12, p.9], [8, §2])

$$\sum_{k=0}^{\infty} F(0, k) = \sum_{n=0}^{\infty} G(n, 0),$$

yielding the theorem with $d_n = D(n-1)$, $L_n = L(n)$, $A_0 = A(0)$, $B_0 = B(0)$, $C_0 = C(0)$. \square

3. PROOF OF THEOREM 2.

To deduce (7) from (8), take $A(0) = C(0) = 0$, $B(0) = 1$ and apply the following formula (see [11, Corollary 2]):

$$(20) \quad \sum_{k=0}^{\infty} F(0, k) = \sum_{n=0}^{\infty} (F(n, n) + G(n, n+1)).$$

Since in this case $L(n) = 0$ for all $n \geq 0$, an easy computation of the right-hand side of (20) by (10), (11), (16)–(18) and substitution (6) lead to the desired conclusion. \square

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